Refinement of the gyrokinetic equations for edge plasmas with large flow shears

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A refined formulation of the gyrokinetic equations for large-flow shears caused by an equilibrium electric field has been presented. It is achieved by choosing more suitable equilibrium drift velocity for the reference frame of a charged particle instead of the previous one [H. Qin, Contrib. Plasma Phys., **46**, 477 (2006)]. This modification yields improvements in the accuracy of the gyrokinetic equations even in the case of considerably large flow. The equations of motion and Maxwell's equations are obtained using the Lie perturbation analysis and the pullback technique. From the numerical comparisons of the gyrokinetic equations given by Qin and the one derived here, the advantage of the present formulation is confirmed for both uniform and nonuniform large electric fields. Parameter dependence of the error in the energy expression is also numerically evaluated. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2902016]

I. INTRODUCTION

Understanding the role of flow shears in turbulent transport is one of the major issues in tokamak plasmas. The stabilizing effect[1–5] of the $\mathbf{E}\times\mathbf{B}$ flow shears on the toroidal ion-temperature-gradient (ITG) mode and various drift waves is believed to be one of the essential elements in the core and edge transport barriers. The existence of large-flow shears associated with the short scale length is a characteristic feature of the edge plasmas. In addition to the relatively short time-scale dynamics such as the microinstabilities, the pedestal plasmas involve the longer time-scale equilibrium dynamics such as the edge localized modes (ELMs) and the Low to High mode transition. In order to treat the multiscale physics such as the microinstabilities and the equilibrium dynamics, a global full-f simulation is required for the understanding of pedestal physics. Recently, the development of such simulation codes[6,7] began. They employ the gyrokinetic equations[8–10] as the fundamental equations to describe the low-frequency behavior of plasmas. The most distinct advantage of the gyrokinetic theory is in the separation of the time scale between the fast gyrating motion of particles and the relatively slow drift motions. Discarding the gyrating motion and the gyrophase dependence of the velocity distribution functions, one can choose a much larger time step than the gyroperiod in simulations. It is also a benefit that the gyroaveraged expressions of the potentials and other physical quantities can reduce the numerical noise caused by the discreteness of the particles and the spatial grids.

The modern derivation of gyrokinetic equations has been developed with the aid of mathematics and the analytic mechanics such as 1-form representations of the particle dynamics, the Lie perturbation analysis, and pullback representations. The commonly used procedures in the derivation are understood as two steps of coordinate transformations and the formulation of Maxwell's equations on the new coordinate. The first transformation introduces the guiding-center position, the gyrophase, and the magnetic moment. The second one decomposes the gyrophase dependences in the 1-form through successive Lie transformations. The gyrophase dependences in the original equations of motion are removed and thus the gyration and the drift motion are decoupled. The Vlasov equation and Maxwell's equations expressed by pullbacks in the new coordinate enable one to treat the low-frequency phenomena without resolving the fast gyrations of particles.

The improvement of the gyrokinetics for the strong $\mathbf{E}\times\mathbf{B}$ drift flow was provided by Littlejohn[11] for the first time and extended for plasma with potential perturbations by Brizard,[12] Hahm,[13] and Qin.[14] Applications to the global linear analysis of ITG modes have also been made.[2,3] We note that the gyrokinetic equations based on the conventional recursive method[15–19] have also been formulated for large-flow shears by Sugama and Horton.[20] Although the formulation by Littlejohn differs slightly from others because of the difference in the expression of mechanics, their basic concepts are the same. They introduced a reference frame moving with the $\mathbf{E}\times\mathbf{B}$ drift velocity in the guiding-center coordinate and decomposed the drift motion and the gyration not in the first-order equations of motion but in the zeroth-order equations. The physical meaning of this treatment is easily understood in an ideal case as follows. If the reference frame is moving with a constant velocity $\mathbf{D}$ and the electromagnetic fields are uniform, the Galilei transformation with $\mathbf{D}$ yields a uniform induced electromotive force, $q\mathbf{v}\times\mathbf{B}$. If the velocity $\mathbf{D}$ is given by the $\mathbf{E}\times\mathbf{B}$ drift velocity, $\mathbf{E}\times\mathbf{B}/B^2$, the perpendicular components of the electric field are canceled, and the particle simply gyrates as if the electric field is not applied from the beginning. In this case, the drift motion is successfully decomposed from the particle motion and included in the zeroth-order equations of motion.

In the case of a general electric field, however, the velocity $\mathbf{D}=\mathbf{E}\times\mathbf{B}/B^2$ acquires the gyrophase dependence through the coupling of the gyration and spatial variation of the potential through $\mathbf{E}=-\nabla\phi(\mathbf{x})$. This dependence makes the derivation of the gyrokinetic equations complicated.[11] On the other hand, if the velocity $\mathbf{D}$ is defined as the $\mathbf{E}\times\mathbf{B}$ drift velocity measured at the guiding-center position as is common in previous works,[12–14] it differs from the averaged drift velocity of the gyrating particle in the case of the nonuniform electric fields. In order to obtain the most appropriate zeroth-order equation of motion, refinement of the velocity of the reference frame is necessary.

In the present study, using the conservation property of the magnetic moment as the criterion of the accuracy of the zeroth-order equation of motion, we examine several kinds of drift velocities including the previous expression and obtain the practically most accurate expression of the velocity $\mathbf{D}$. The advantages of our gyrokinetic equations of motion are verified through comparisons between the numerical solutions of the previous gyrokinetic equations and ours.

In Sec. II, the guiding-center coordinate variables are introduced in the 1-form for a single particle. An equilibrium drift velocity $\mathbf{D}$ is also introduced in the transformation of the velocity coordinates so that the particle motion becomes a nearly simple gyration in the reference frame. In Sec. III, the criterion to choose the appropriate velocity $\mathbf{D}$ is discussed. After examining several possible choices for $\mathbf{D}$, a most practical expression is determined. The rest of the standard procedures—Lie perturbation analysis and the formulation of field equations—are carried out and the gyrokinetic equations are obtained in Sec. IV. The accuracy of the resultant equations is compared with those of Qin's formulation[14] in Sec. V. Finally, conclusions are presented in Sec. VI.

II. PRELIMINARY TRANSFORMATION

The first step in the derivation of gyrokinetic equations is a guiding-center transformation introducing a guiding-center position $\mathbf{X}$, a gyrophase Θ, a perpendicular velocity $V_\perp$, and a parallel velocity $V_\parallel$. In the conventional derivations,[8–10] the velocity is simply separated into the perpendicular and parallel components, i.e., $v_\perp \equiv |\hat{\mathbf{b}}\times(\mathbf{v}\times\hat{\mathbf{b}})|$ and $v_\parallel \equiv \mathbf{v}\cdot\hat{\mathbf{b}}$, where the unit vector $\hat{\mathbf{b}}$ represents the magnetic field direction. In the gyrokinetic theory for large $\mathbf{E}\times\mathbf{B}$ drift flow shears,[11–14] however, the velocity space is defined on a reference frame moving with an equilibrium $\mathbf{E}\times\mathbf{B}$ velocity $\mathbf{D}$. The vector field of the flow plays an essential role in the improvement of the theory and is discussed in Sec. III. In order to distinguish the modified velocity space variables from those on the stationary frame, $v_\perp$ and $v_\parallel$, we denote the new velocity components in the moving frame as capital letters, $V_\perp$ and $V_\parallel$. We note that if the velocity $\mathbf{D}$ is zero everywhere, the modified gyrokinetic theory coincides with that of usual ordering, i.e., the equilibrium flow is much slower than the thermal velocity. In this case, the guiding-center velocity variables also coincide, or $V_\perp = v_\perp$ and $V_\parallel = v_\parallel$.

We assume that the equilibrium $\mathbf{E}\times\mathbf{B}$ flow is a function of the guiding-center position $\mathbf{X}$ to avoid undesirable complexities due to the dependences on the velocity space such as $\partial\mathbf{D}/\partial V_\parallel$. The guiding-center transformation is defined as inverse coordinate transformations,

$$\mathbf{x} \equiv \mathbf{X} + \frac{mV_\perp}{qB(\mathbf{X})}\hat{\mathbf{a}}'(\mathbf{X},\Theta'), \tag{1}$$

$$\mathbf{v} \equiv \mathbf{D}(\mathbf{X}) + V_\perp\hat{\mathbf{c}}'(\mathbf{X},\Theta') + V_\parallel\hat{\mathbf{b}}(\mathbf{X}), \tag{2}$$

where the gyrophase Θ', the orthonormal vectors $\hat{\mathbf{b}} \equiv \mathbf{B}/B$, $\hat{\mathbf{c}} \equiv \hat{\mathbf{e}}_1 \cos\Theta' - \hat{\mathbf{e}}_2 \sin\Theta'$, and $\hat{\mathbf{a}} \equiv \hat{\mathbf{e}}_1 \sin\Theta' + \hat{\mathbf{e}}_2\cos\Theta'$ are introduced. The perpendicular unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are functions of the guiding-center position $\mathbf{X}$ and assumed to be given beforehand. Since their definitions are arbitrary unless they have any singularities, the gyrogauge transformation,[8,11] $\Theta \equiv \Theta' + \varphi$, is introduced to remove the arbitrariness in the definition of the gyrophase Θ'. The gyrogauge φ is given by $\varphi \equiv \int_0^t [(d\mathbf{X}/dt)\cdot\nabla\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 + (\partial\hat{\mathbf{e}}_1/\partial t)\cdot\hat{\mathbf{e}}_2]dt$. We denote the new direction vectors $\hat{\mathbf{c}}(\mathbf{X},\Theta) \equiv \hat{\mathbf{c}}(\mathbf{X},\Theta-\varphi)$ and $\hat{\mathbf{a}}(\mathbf{X},\Theta) \equiv \hat{\mathbf{a}}(\mathbf{X},\Theta-\varphi)$ simply by $\hat{\mathbf{c}}$ and $\hat{\mathbf{a}}$ in the remainder of this paper. We note that if the base direction for Θ is defined by $\hat{\mathbf{u}}(\mathbf{X}) \equiv \hat{\mathbf{c}}(\mathbf{X},0)$, its time evolution is described by the "rotationless" transport equation,[21] $d\hat{\mathbf{u}}/dt = -\hat{\mathbf{b}}(d\hat{\mathbf{b}}/dt)\cdot\hat{\mathbf{u}}$. The usage of the new gyrophase Θ ensures the uniqueness of the base direction for the gyrophase Θ. The relation $d\hat{\mathbf{a}}\cdot\hat{\mathbf{c}} = d\mathbf{X}\cdot\nabla\hat{\mathbf{a}}\cdot\hat{\mathbf{c}} + dt(\partial\hat{\mathbf{a}}/\partial t)\cdot\hat{\mathbf{c}} = d\Theta$ is utilized later in the gauge transformation of 1-form to simplify the calculations.

Although the definition of the guiding-center $\mathbf{X}$ does not have an explicit dependence on $\mathbf{D}$, a difference arises from the modification of the perpendicular velocity $V_\perp$. When the guiding-center position for $\mathbf{D}=0$ is denoted by $\mathbf{X}'$, the difference from the present guiding-center position is written as $\Delta_\mathrm{p} \equiv \mathbf{X} - \mathbf{X}' \simeq (m/qB)\hat{\mathbf{b}}\times\mathbf{D}$, where we used the approximation $B(\mathbf{X}) \simeq B(\mathbf{X}')$. If the velocity $\mathbf{D}$ is given by the simple $\mathbf{E}\times\mathbf{B}$ drift velocity, the quantity Δ_p is reduced to $\Delta_p = -m\nabla_\perp\phi/qB^2$. The fact that its derivative with respect to time coincides with the polarization drift velocity, $\mathbf{v}_p \equiv -(d/dt)m\nabla_\perp\phi/qB^2$, indicates that the modified guiding-center coordinate $\mathbf{X}$ recovers the polarization drift due to the equilibrium electric fields in the zeroth-order equations of motion, i.e., $d\mathbf{X}/dt - d\mathbf{X}'/dt = \mathbf{v}_p$.

In order to separate the fundamental 1-form for a single charged particle,

$$\gamma \equiv [q\mathbf{A}(\mathbf{x}) + m\mathbf{v}]\cdot d\mathbf{x} - \left[\frac{m}{2}v^2 + q\phi(\mathbf{x})\right]dt, \tag{3}$$

into zeroth-, first-, and successive higher-order components, we introduce the perturbation potentials $\phi = \phi_0 + \phi_1$ and $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$ and the following orderings:

$$\phi_1 \sim \epsilon\frac{mv_t^2}{q}, \quad \mathbf{A}_1 \sim \epsilon\frac{mv_t}{q}, \tag{4}$$

where the quantity m, q, and v_t are mass, charge, and thermal speed of the particle species, respectively. The frequency of the perturbation ω is assumed to be much lower than the gyrofrequency Ω; $\omega \sim \epsilon\Omega$. The equilibrium $\mathbf{E}\times\mathbf{B}$ drift speed is assumed to be comparable to the thermal speed at most, and the spatial scale of equilibrium magnetic field is assumed to be a second-order quantity,

$$\frac{E_0}{B_0} \sim v_t, \quad \frac{|\nabla B_0|}{B_0} \sim \epsilon^2 \frac{v_t}{\Omega}. \tag{5}$$

The time scale of the equilibrium potentials is assumed as

$$\frac{\partial \phi_0}{\partial t} \sim \epsilon^2 v_t^2 B_0, \quad \frac{\partial \mathbf{A}_0}{\partial t} \sim \mathbf{0}. \tag{6}$$

From the fundamental 1-form, Eq. (3), and the above orderings, the 1-form in the guiding-center coordinate can be written order by order,

$$\gamma_0 = (q\mathbf{A}_0 + mV_\parallel \hat{\mathbf{b}} + m\mathbf{D}) \cdot d\mathbf{X} + \frac{m}{q}\mu d\Theta - \left[\frac{m}{2}V_\parallel^2 + B_0\mu + \frac{m}{2}D^2 + q\bar{\phi}_0\right]dt, \tag{7}$$

$$\gamma_1 = (q\mathbf{A}_1 - m\rho \nabla \mathbf{D} \cdot \hat{\mathbf{a}}) \cdot d\mathbf{X} + q\rho \mathbf{A}_1 \cdot \hat{\mathbf{c}} d\Theta + \frac{1}{V_\perp}\mathbf{A}_1 \cdot \hat{\mathbf{a}} d\mu - [q\tilde{\phi}_0 + mV_\perp \mathbf{D} \cdot \hat{\mathbf{c}} + q\phi_1(\mathbf{X} + \rho\hat{\mathbf{a}})]dt, \tag{8}$$

$$\gamma_2 = \left[\frac{m\mu}{q}(\hat{\mathbf{a}} \cdot \nabla\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}\hat{\mathbf{b}} - \hat{\mathbf{a}} \cdot \nabla \ln B_0 \hat{\mathbf{c}}) - \frac{mV_\perp V_\parallel}{\Omega}\nabla \hat{\mathbf{b}} \cdot \hat{\mathbf{a}}\right] \cdot d\mathbf{X} - \frac{m\mu}{q}\rho\hat{\mathbf{a}} \cdot \nabla \ln B_0 d\Theta - m\rho\frac{\partial \mathbf{D}}{\partial t} \cdot \hat{\mathbf{a}} dt. \tag{9}$$

The higher-order components, $\gamma_3, \gamma_4, \ldots$, are omitted. Gyroaveraged quantities and their corresponding perturbation components for the electromagnetic potentials, $\psi = \phi_0$, ϕ_1, or $\mathbf{A}_1$, are denoted by $\bar{\psi} \equiv \int \psi(\mathbf{X}+\rho\hat{\mathbf{a}})d\Theta/2\pi$ and $\tilde{\psi} \equiv \psi(\mathbf{X}+\rho\hat{\mathbf{a}}) - \bar{\psi}$. Here the perturbation of the equilibrium potential is expressed as $\tilde{\phi}_0$ instead of the Taylor expanded one, $\tilde{\phi}_0 \simeq \rho\hat{\mathbf{a}} \cdot \nabla\phi_0$. Although the latter form is commonly used in previous works,[12–14] the former exact form without the Taylor expansion is desirable in the case of the present ordering, $E_0/B_0 \sim v_t$. The gyroradius and the direction of the equilibrium magnetic field at $\mathbf{X}$ are denoted by $\rho \equiv mV_\perp/qB_0(\mathbf{X})$ and $\hat{\mathbf{b}} \equiv \mathbf{B}_0/B_0$, respectively. The magnetic moment $\mu \equiv mV_\perp^2/B_0$ has been introduced and the gauge transformation, $\gamma \mapsto \gamma + dS$, has been applied to simplify the expressions.

From the truncated zeroth-order 1-form under the assumption of the drift-kinetic ordering $k_\perp \rho \sim \epsilon \ll 1$,

$$\gamma_{\text{drift}} = (q\mathbf{A}_0 + mV_\parallel \hat{\mathbf{b}} + m\mathbf{D}) \cdot d\mathbf{X} + \frac{m}{q}\mu d\Theta - \left[\frac{m}{2}V_\parallel^2 + B_0\mu + \frac{m}{2}D^2 + q\phi_0\right]dt, \tag{10}$$

we can obtain the zeroth-order drift-kinetic equations,

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_\parallel^*}\left[\mathbf{B}^* V_\parallel + \hat{\mathbf{b}} \times \left(\nabla\phi_0 + \frac{\mu}{q}\nabla B_0 + \frac{m}{2q}\nabla D^2 + \frac{m}{q}\frac{\partial \mathbf{D}}{\partial t}\right)\right], \tag{11}$$

$$\frac{d\Theta}{dt} = \frac{qB_0}{m}, \tag{12}$$

$$\frac{d\mu}{dt} = 0, \tag{13}$$

$$\frac{dV_\parallel}{dt} = -\frac{q\mathbf{B}^*}{mB_\parallel^*} \cdot \left(\nabla\phi_0 + \frac{\mu}{q}\nabla B_0 + \frac{m}{2q}\nabla D^2 + \frac{m}{q}\frac{\partial \mathbf{D}}{\partial t}\right), \tag{14}$$

where a modified magnetic field is introduced as

$$\mathbf{B}^* \equiv \nabla \times \left(\mathbf{A}_0 + \frac{mV_\parallel}{q}\hat{\mathbf{b}} + \frac{m}{q}\mathbf{D}\right), \tag{15}$$

$$B_\parallel^* \equiv \hat{\mathbf{b}} \cdot \mathbf{B}^* = B_0 + \frac{mV_\parallel}{q}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \frac{m}{q}\hat{\mathbf{b}} \cdot \nabla \times \mathbf{D}. \tag{16}$$

We confirm that the velocity of the guiding center involves $\mathbf{E}\times\mathbf{B}$, grad B, curvature, and polarization drifts due to the temporal variation of ϕ_0 and due to the parallel motion by taking account of the relation $\mathbf{B}^* = B_\parallel^* \hat{\mathbf{b}} + (mV_\parallel/q)\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}) + (m/q)\nabla \times \mathbf{D}|_\perp$. Although we omitted the Baños drift[22] in the above equations, it can be recovered from the second-order 1-form. The phase-space volume is calculated as $B_\parallel^*/m$ and Liouville's theorem is confirmed; $(\partial/\partial Z^i)(B_\parallel^* \dot{Z}^i/m) = 0$, where the coordinate variables, $(t, \mathbf{X}, \Theta, \mu, V_\parallel)$, are denoted by Z^i for $i = 0, 1, \ldots, 6$.

III. EQUILIBRIUM DRIFT VELOCITY

In this section, we discuss how the equilibrium drift velocity $\mathbf{D}$ should be chosen. The introduction of the vector field $\mathbf{D}$ in Sec. II is aimed at decomposing the circular gyration from the particle dynamics. Therefore, one might expect the drift velocity obtained from the drift-kinetic equation (11) to be the best choice of $\mathbf{D}$. The dependences on the velocity space, however, cause difficulties in the calculations, i.e., the fundamental 1-form acquires some additional terms such as $\partial\mathbf{D}/\partial\mu$ and $\partial\mathbf{D}/\partial V_\parallel$. In order to keep the complexities in the same level as the previous study by Qin,[14] we assume that the vector field $\mathbf{D}$ is an only function of the guiding-center position $\mathbf{X}$. From the guiding-center velocity, Eq. (11), and this assumption, we can obtain the practically most precise drift velocity,

$$\mathbf{D} \equiv \frac{\hat{\mathbf{b}}}{1 + \hat{\mathbf{b}} \cdot \nabla \times \mathbf{D}/\Omega} \times \left(\frac{\nabla\phi_0(\mathbf{X})}{B_0} + \frac{\nabla D^2}{2\Omega}\right). \tag{17}$$

This nonlinear differential vector equation can be rewritten as another mathematically equivalent form,

$$\mathbf{D} \equiv \hat{\mathbf{b}} \times \left(\frac{\nabla \phi_0}{B_0} + \mathbf{D} \cdot \frac{\nabla \mathbf{D}}{\Omega} \right). \tag{18}$$

Since the definition in the previous study is given by $\mathbf{D}_{\mathrm{Qin}} \equiv \hat{\mathbf{b}} \times \nabla \phi_0 / B$, the modification introduced into our definition is the second term on the right-hand side in Eq. (18).

The physical meaning of Eq. (18) is easily understood through the vector product with the magnetic field,

$$m\mathbf{D} \cdot \nabla \mathbf{D}\big|_{\perp} = q\mathbf{D} \times \mathbf{B}_0 - q\nabla_{\perp} \phi_0. \tag{19}$$

This equation represents the perpendicular force balance in the stationary flow. The convection term on the left-hand side is missing in $\mathbf{D}_{\mathrm{Qin}}$. A similar equation, $\mathbf{u} \cdot \nabla \mathbf{u} = \Omega \mathbf{u} \times \hat{\mathbf{b}} - (\Omega / B_0) \nabla \phi_0 - \nabla P / mN$, is discussed by Brizard[12] in the formulation of the gyrokinetic Vlasov equation for the plasma with toroidal rotation. Since their main interest is in the toroidal flow, their equilibrium velocity includes the parallel flow and they adopt an approximate expression of the flow, $\mathbf{u}_0 = u_{0\|} \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \nabla \phi_0 / B_0$, as the vector field $\mathbf{D}$. Although the equilibrium parallel flow can be included in the definition of $\mathbf{D}$, we omit it for clarity in the examination of the equilibrium perpendicular drift velocity.

One of the straightforward ways to examine the properness of this choice is to verify the conservation property of the magnetic moment μ. Since the gyrophase dependence is truncated in the drift-kinetic 1-form, Eq. (10), the drift-kinetic equation (13) conserves the magnetic moment. The general equation of motion, however, does not conserve μ,

$$\frac{d\mu}{dt} = qV_{\perp} \left(\mathbf{D} \cdot \hat{\mathbf{a}} - \hat{\mathbf{c}} \cdot \frac{\nabla \phi_0}{B_0} - \dot{X} \cdot \frac{\nabla \mathbf{D}}{\Omega} \cdot \hat{\mathbf{c}} \right), \tag{20}$$

because of the gyrophase dependence in the general 1-form

$$\gamma = (q\mathbf{A}_0 + mV_{\|} \hat{\mathbf{b}} + m\mathbf{D} - m\rho \nabla \mathbf{D} \cdot \hat{\mathbf{a}}) \cdot d\mathbf{X} + \frac{m\mu}{q} d\Theta - \left[\frac{m}{2} V_{\|}^2 + B\mu + \frac{m}{2} D^2 + mV_{\perp} \mathbf{D} \cdot \hat{\mathbf{c}} + q\phi_0(\mathbf{X} + \rho \hat{\mathbf{a}}) \right] dt, \tag{21}$$

where we neglect ∇B_0, $\nabla \hat{\mathbf{b}}$, and the perturbation potentials. If the gyration and the drift motion are decoupled well, the value of $d\mu/dt$ should be small. We calculate $d\mu/dt$ for three choices of $\mathbf{D}$. The first is the zero velocity case, which corresponds to the conventional formulation with the equilibrium potential ϕ_0 but without special treatments for the large flow. The time derivative of μ becomes

$$\frac{d\mu}{dt} = -\frac{qV_{\perp}}{B} \hat{\mathbf{c}} \cdot \nabla \phi_0. \tag{22}$$

If the electric field is large, $E_0 / B_0 \sim v_t$, the variation of the magnetic moment becomes the same order as μ itself. In other words, the electric field has to be as small as the perturbation potential ϕ_1 in this case. The second is the simple $\mathbf{E} \times \mathbf{B}$ drift velocity, which corresponds to Qin's formulation,

$$\frac{d\mu}{dt} = -\frac{qV_{\perp}}{B} \hat{\mathbf{c}} \cdot \nabla [\phi_0 - \phi_0(\mathbf{X})] - qV_{\perp} \dot{\mathbf{X}} \hat{\mathbf{a}} : \frac{\nabla \nabla \phi_0(\mathbf{X})}{B}. \tag{23}$$

Since most of the electric field is canceled, it is applicable for a strong electric field in this case. The second derivative of ϕ, however, appears in $d\mu/dt$ and can be significant if the potential contour has a large curvature. The last is our definition, Eq. (18),

$$\frac{d\mu}{dt} = -qV_{\perp} \left[\frac{\nabla(\phi_0 - \phi_0(\mathbf{X}))}{B} + (\dot{X} - \mathbf{D}) \cdot \frac{\nabla \mathbf{D}}{\Omega} \right] \cdot \hat{\mathbf{c}}. \tag{24}$$

Since the velocity of the guiding center $\dot{X}$ can be approximated as $\dot{X} \simeq V_{\|} \mathbf{B}^* / B_{\|}^* + \mathbf{D}$, the second term proportional to $\nabla \mathbf{D}$ is considerably reduced. From the above observations, we confirm that the refinement of the gyrokinetic equations is achieved through our new choice of the vector field, Eq. (18). Numerical verifications of the new equilibrium velocity are given in Sec. V.

IV. GYROKINETIC EQUATIONS

A. The general derivation of the gyrokinetic equations

The remaining procedures to obtain the gyrokinetic equations are the Lie perturbation analysis and the formulation of the gyrokinetic Maxwell's equations. Since these treatments are essentially the same as the previous works,[8–14] we omit detailed discussions and describe the outline and the results.

Successive Lie transformations are introduced, $\mathcal{T} \equiv \cdots \exp(\epsilon^2 \mathcal{L}_2) \exp(\epsilon \mathcal{L}_1)$. The ith order operator $\mathcal{L}_i$ is a Lie derivative operator defined by an ith order Lie generator g_i, $\mathcal{L}_i v \equiv i_{g_i}(dv)$. Although the correct Lie derivative has the form $i_g(dv) + d(i_g v)$, we adopt the truncated expression because the second term does not affect scalars and resultant equations of motion. In other words, the second term $d(i_g v)$ is eliminated through the gauge transformation. The guiding-center coordinate variables $Z^i = (t, \mathbf{X}, \Theta, \mu, V_{\|})$ are transformed to the gyrocenter coordinate variables $\bar{Z}^i = (t, \bar{\mathbf{X}}, \bar{\Theta}, \bar{\mu}, \bar{V}_{\|})$, $\bar{Z} = \mathcal{T} Z = \cdots \exp(\epsilon^2 i_{g_2} d) \exp(\epsilon i_{g_1} d) Z$, where the time variable t is not changed through the Lie transformation. The 1-form in the guiding-center coordinate, $\gamma = \gamma_0 + \epsilon \gamma_1 + \epsilon^2 \gamma_2 + \cdots$, is also transformed to the gyrokinetic 1-form, $\Gamma = \Gamma_0 + \epsilon \Gamma_1 + \epsilon^2 \Gamma_2 + \cdots$, in the gyrocenter coordinate. The new 1-form Γ is determined by the Lie generator g_i and the gauge function S_i,

$$\Gamma_0 = \gamma_0, \tag{25}$$

$$\Gamma_1 = \gamma_1 - i_{g_1} d\gamma_0 + dS_1, \tag{26}$$

$$\Gamma_2 = \gamma_2 - i_{g_1} d\gamma_1 + \tfrac{1}{2}[(i_{g_1} d)^2 - i_{g_2} d]\gamma_0 + dS_2. \tag{27}$$

The first-order gauge function S_1 and the first-order Lie generator g_1 are determined as follows:

$$\mathcal{V}_0^i \frac{\partial S_1}{\partial \bar{Z}^i} = -\mathcal{V}_0^i(\gamma_{1i} - \langle \gamma_{1i} \rangle), \tag{28}$$

$$g_1^j = \sigma^{ij}\left(\gamma_{1i} + \frac{\partial S_1}{\partial \bar{Z}^i}\right) \quad \text{for } j \neq 0, \tag{29}$$

where the vector field $\mathcal{V}_0^i$ is defined as the flow created by the zeroth-order equation of motion,

$$\mathcal{V}_0^i = \sigma^{ij}\left(\frac{\partial \gamma_{0j}}{\partial \bar{Z}^0} - \frac{\partial \gamma_{00}}{\partial \bar{Z}^j}\right). \tag{30}$$

The time-component of the Lie generator, g_1^0, is defined as zero, which corresponds to the identical transformation for the variable t. The tensor $\vec{\sigma}$ represents the Poisson tensor calculated from the zeroth-order 1-form, Eq. (7),

$$\vec{\sigma} = \left(\begin{array}{c|ccc} \hat{\mathbf{b}} \times \vec{I}/qB_\parallel^* & \mathbf{0} & \mathbf{0} & \mathbf{B}^*/mB_\parallel^* \\ \hline \mathbf{0} & 0 & q/m & 0 \\ \mathbf{0} & -q/m & 0 & 0 \\ -\mathbf{B}^*/mB_\parallel^* & 0 & 0 & 0 \end{array}\right). \tag{31}$$

The gyrokinetic 1-form is obtained up to the first order,

$$\Gamma = \Gamma_0 + \Gamma_1 = \gamma_0 - \mathcal{V}_0^i \langle \gamma_{1i} \rangle dt. \tag{32}$$

This 1-form yields the gyrokinetic equations of motion,

$$\frac{dZ^i}{dt} = \sigma^{ij}\left(\frac{\partial \Gamma_j}{\partial t} - \frac{\partial \Gamma_0}{\partial Z^j}\right). \tag{33}$$

The Vlasov equation and its conservation form are also obtained as $\dot{Z}^i(\partial \bar{F}/\partial Z^i)=0$ and $(\partial/\partial Z^i)(\dot{Z}^i B_\parallel^* \bar{F})=0$, respectively.

The gyrokinetic expressions of Maxwell's equations can be obtained by writing the charge density and current density with the distribution function in the gyrocenter coordinate. The formulation is achieved by the pullback technique introduced by Brizard[8] and Qin.[9,23] When a physical quantity is given by $\lambda(\mathbf{x},\mathbf{v})$, e.g., $\lambda=q$ for the charge density and $\lambda=q\mathbf{v}$ for the current density, its moment is expressed as $\bar{\lambda}(\mathbf{x})=\int \lambda(\mathbf{x}',\mathbf{v}')f(\mathbf{x}',\mathbf{v}')\delta(\mathbf{x}'-\mathbf{x})d^3x'd^3v'$. Thus, the pullback expression of λ, $\Lambda(Z)=\lambda(\mathbf{X}+\rho\hat{\mathbf{a}},\mathbf{v}(\Theta,\mu,V_\parallel))$yields the averaged quantity in the guiding-center coordinate, $\bar{\lambda}(\mathbf{x})=\int \Lambda(Z')F(Z')\delta(\mathbf{X}'+\rho\hat{\mathbf{a}}-\mathbf{x})B_\parallel^*/md^6Z'$. The distribution function $F(Z)$ is that of the guiding center and can also be written in the gyrocenter coordinate as $f(z)=F(Z)=\bar{F}(\bar{Z})$. The gyrocenter distribution function $\bar{F}$ is usually assumed to be independent of the gyrophase $\bar{\Theta}$. From the pullback expressions, we can write the moment integral of arbitrary physical quantities with the gyrocenter distribution function $\bar{F}$,

$$\bar{\lambda}(\mathbf{x}) = \int \Lambda(Z')\mathcal{T}^*\bar{F}(Z')\delta(\mathbf{X}'+\rho\hat{\mathbf{a}}-\mathbf{x})\frac{B_\parallel^*}{m}d^6Z'. \tag{34}$$

B. Limiting case with electrostatic perturbation

We show the gyrokinetic equations with the electrostatic perturbation as an example of the limiting case. The equations of motion in this section are used in Sec. V for numerical verifications. We use some approximations commonly assumed in the analysis of the microinstabilities.[13,24,25] First, the vector field of the zeroth order, $\mathcal{V}_0$, used in the determining equations of the gauge function S_1, Eq. (28), and the 1-form, Eq. (32), is reduced to $\mathcal{V}_0 \simeq (\bar{V}_\parallel\hat{\mathbf{b}}+\mathbf{D})\partial_{\bar{\mathbf{X}}}+\Omega\partial_{\bar{\Theta}}+\partial_t$. Second, we assume that the dependences of the gauge function S_1 on the coordinate variables $(\bar{\mathbf{X}},\bar{\mu},\bar{V}_\parallel,t)$ are much smaller than on the gyrophase $\bar{\Theta}$, i.e., $dS_1 \simeq \partial_{\bar{\Theta}}S_1 d\bar{\Theta}$.

Under these approximations, the gyrokinetic 1-form is obtained up to the first order as

$$\Gamma = (q\mathbf{A}_0 + m\bar{V}_\parallel\hat{\mathbf{b}} + m\mathbf{D})\cdot d\bar{\mathbf{X}} + \frac{m}{q}\bar{\mu}d\bar{\Theta} - \left[\frac{m}{2}\bar{V}_\parallel^2 + B\bar{\mu} + \frac{m}{2}D^2 + q\bar{\phi}_0 + q\bar{\phi}_1\right]dt. \tag{35}$$

Although this equation is almost same as the corresponding equations in the previous works of Hahm[13] and Qin,[14] there are two differences. One is the definition of $\mathbf{D}$ from the simple $\mathbf{E}\times\mathbf{B}$ drift velocity to the generalized one. The other difference is the expression of the gyroaveraged equilibrium potential ϕ_0. Hahm and Qin employ $\phi_0+(m\mu/2q^2)\hat{\mathbf{b}}\cdot\nabla\times\mathbf{D}$ and $\phi_0+(\mu/2q\Omega)\nabla_\perp^2\phi_0$ as the gyroaveraged potential, respectively. The term $(m\mu/2q^2)\hat{\mathbf{b}}\cdot\nabla\times\mathbf{D}$ is also found in Brizard's paper[12] and can be written as $(\mu/2q\Omega)\nabla_\perp^2\phi_0$ approximately. Since our expression of the gyroaveraged potential is also approximated as $\bar{\phi}_0 \simeq \phi_0+(\mu/2q\Omega)\nabla_\perp^2\phi_0$, these three expressions are essentially the same. Our equation, however, has an advantage in the rigorousness because of the absence of the truncation due to the Taylor expansion.

The first-order gauge function is also obtained as

$$S_1 = \frac{q}{\Omega}\tilde{\Phi}_0 + \frac{q\rho}{\Omega}\hat{\mathbf{c}}\cdot\nabla\phi_0 - \frac{m\bar{V}_\parallel\rho}{\Omega}\hat{\mathbf{b}}\cdot\nabla\mathbf{D}\cdot\hat{\mathbf{c}} + \frac{q}{\Omega}\tilde{\Phi}_1, \tag{36}$$

where the notation $\tilde{\Phi} \equiv \oint \tilde{\phi}d\bar{\Theta} - \langle \oint \tilde{\phi}d\bar{\Theta}\rangle_{\bar{\Theta}}$ is introduced. It is calculated by the partial integral,

$$\tilde{\Phi} = \frac{1}{2\pi}\int_{\Theta}^{\Theta+2\pi}(\Theta'-\Theta-\pi)\phi[\mathbf{X}+\rho\hat{\mathbf{a}}(\Theta')]d\Theta'. \tag{37}$$

The previous expression given by Qin for the simple equilibrium velocity $\mathbf{D}_{\mathrm{Qin}} \equiv \hat{\mathbf{b}}\times\nabla\phi_0/B$ is

$$S_{1\mathrm{Qin}} = \frac{q}{\Omega}\tilde{\Phi}_0 + \frac{q\rho}{\Omega}\hat{\mathbf{c}}\cdot\nabla\phi_0 - \frac{V_\perp}{B^2}\mathbf{D}_{\mathrm{Qin}}\cdot\nabla\mathbf{D}_{\mathrm{Qin}}\cdot\hat{\mathbf{c}} - \frac{V_\perp V_\parallel}{B^2}\hat{\mathbf{b}}\cdot\nabla\mathbf{D}_{\mathrm{Qin}}\cdot\mathbf{c} + \frac{q}{\Omega}\tilde{\Phi}_1. \tag{38}$$

The third term in $S_{1\mathrm{Qin}}$ has been canceled in our gauge function S_1. This fact shows that the zeroth-order equation of motion adopted here is more accurate than that of Qin's study. The first-order Lie generator g_1 is calculated from this approximated S_1 as

$$g_1^{\bar{\mathbf{X}}} = -\frac{1}{qB_\parallel^*}\hat{\mathbf{b}} \times \Bigg[-m\rho \nabla \mathbf{D} \cdot \hat{\mathbf{a}} + \nabla\left(\frac{q}{\Omega}\tilde{\Phi}_0 + \frac{q}{\Omega}\tilde{\Phi}_1 + \frac{q\rho}{\Omega}\hat{\mathbf{c}} \cdot \nabla \phi_0 - \frac{mV_\parallel \rho}{\Omega}\hat{\mathbf{b}} \cdot \nabla \mathbf{D} \cdot \hat{\mathbf{c}}\right)\Bigg] - \frac{\mathbf{B}^*}{mB_\parallel^*}\left(-\frac{m\rho}{\Omega}\hat{\mathbf{b}} \cdot \nabla \mathbf{D} \cdot \hat{\mathbf{c}}\right), \tag{39}$$

$$g_1^{\bar{\Theta}} = -\frac{q^2}{m\Omega}\left(\frac{\partial \tilde{\Phi}_0}{\partial \bar{\mu}} + \frac{\partial \tilde{\Phi}_1}{\partial \bar{\mu}} + \frac{1}{qV_\perp}\hat{\mathbf{c}} \cdot \nabla \phi_0 - \frac{mV_\parallel}{q^2 V_\perp}\hat{\mathbf{b}} \cdot \nabla \mathbf{D} \cdot \hat{\mathbf{c}}\right), \tag{40}$$

$$g_1^{\bar{\mu}} = \frac{q^2}{m\Omega}\left(\tilde{\phi}_0 + \tilde{\phi}_1 - \rho \hat{\mathbf{a}} \cdot \nabla \phi_0 + \frac{m\rho V_\parallel}{q}\hat{\mathbf{b}} \cdot \nabla \mathbf{D} \cdot \hat{\mathbf{a}}\right), \tag{41}$$

$$g_1^{\bar{V}_\parallel} = \frac{\mathbf{B}^*}{mB_\parallel^*} \cdot \Bigg[-m\rho \nabla \mathbf{D} \cdot \hat{\mathbf{a}} + \nabla\left(\frac{q}{\Omega}\tilde{\Phi}_0 + \frac{q}{\Omega}\tilde{\Phi}_1 + \frac{q\rho}{\Omega}\hat{\mathbf{c}} \cdot \nabla \phi_0 - \frac{mV_\parallel \rho}{\Omega}\hat{\mathbf{b}} \cdot \nabla \mathbf{D} \cdot \hat{\mathbf{c}}\right)\Bigg]. \tag{42}$$

Terms proportional to $\mathbf{D} \cdot \nabla \mathbf{D}$ have been canceled also in the Lie generator.

Since the first-order equations have been obtained from the above discussions, we can obtain the second-order 1-form as follows:

$$\Gamma_2 = i_{\mathcal{V}_0}\left\langle \gamma_2 - i_{g1} d\gamma_1 + \frac{1}{2}(i_{g1}d)^2 \gamma_0 \right\rangle dt. \tag{43}$$

This quantity has only the second-order terms in magnitude but consists of linear and nonlinear, or square, terms with respect to the perturbation potential ϕ_1. If we omit the linear terms, the explicit nonlinear part of Γ_2 under the assumption of electrostatic perturbation is given by

$$\Gamma_2 \simeq \frac{q^3}{2m\Omega}\frac{\partial}{\partial \bar{\mu}}\langle (\tilde{\phi})^2 \rangle dt + \frac{q}{2B_\parallel^* \Omega}(\hat{\mathbf{b}} \times \nabla \tilde{\phi}_1) \cdot \nabla \tilde{\Phi}_1 dt. \tag{44}$$

This expression coincide with that of Hahm's equation,[13,24] Although we omitted the linear terms here, they may become significant in the presence of a strong perturbation potential.

From the gyrokinetic 1-form, the gyrokinetic equations of motion are obtained as

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_\parallel^*}\Bigg[\mathbf{B}^* V_\parallel + \hat{\mathbf{b}} \times \left(\nabla \bar{\phi}_0 + \nabla \bar{\phi}_1 + \frac{\mu}{q}\nabla B_0 + \frac{m}{2q}\nabla D^2 + \frac{m}{q}\frac{\partial \mathbf{D}}{\partial t}\right)\Bigg], \tag{45}$$

$$\frac{d\Theta}{dt} = \frac{qB_0}{m} + \frac{q^2}{m}\frac{\partial \bar{\phi}_0}{\partial \bar{\mu}} + \frac{q^2}{m}\frac{\partial \bar{\phi}_1}{\partial \bar{\mu}}, \tag{46}$$

$$\frac{d\mu}{dt} = 0, \tag{47}$$

$$\frac{dV_\parallel}{dt} = -\frac{q\mathbf{B}^*}{mB_\parallel^*} \cdot \left(\nabla \bar{\phi}_0 + \nabla \bar{\phi}_1 + \frac{\mu}{q}\nabla B_0 + \frac{m}{2q}\nabla D^2 + \frac{m}{q}\frac{\partial \mathbf{D}}{\partial t}\right), \tag{48}$$

where the second-order nonlinear term was omitted here. The term proportional to $\partial \mathbf{D}/\partial t$ in Eq. (45) represents the polarization drift due to the temporal variation of the equilibrium electric field. This term is not in Hahm's paper[13] because of the assumption that the equilibrium potential is constant in time. In the case of numerical calculations, it is convenient to rewrite the $\bar{\mu}$ derivative of the gyroaveraged potential expressed as $\partial \bar{\phi}/\partial \bar{\mu} = \langle \hat{\mathbf{a}} \cdot \nabla \phi(\mathbf{X}+\rho \hat{\mathbf{a}}) \rangle / q\bar{V}_\perp = \langle \hat{\mathbf{c}}\hat{\mathbf{c}} : \nabla\nabla \phi(\mathbf{X}+\rho \hat{\mathbf{a}}) \rangle / B$. If the spatial scale of the equilibrium potential is much larger than the gyroradius, we can use approximate expressions, $\bar{\phi}_0 \simeq \phi_0 + (\rho^2/4)\nabla_\perp^2 \phi_0$ and $\partial \bar{\phi}_0/\partial \bar{\mu} \simeq \nabla_\perp^2 \phi_0 / 2B$.

The particle density is calculated up to the first order as a pullback expression,

$$n(\mathbf{x}) = \int \left(\bar{F} + g_1^{\bar{\mathbf{X}}} \cdot \frac{\partial \bar{F}}{\partial \mathbf{X}} + g_1^{\bar{\mu}}\frac{\partial \bar{F}}{\partial \mu} + g_1^{\bar{V}_\parallel}\frac{\partial \bar{F}}{\partial V_\parallel}\right) \times \frac{B_\parallel^*}{m}\delta(\mathbf{X} + \rho \hat{\mathbf{a}} - \mathbf{x}) d^3 X d\Theta d\mu dV_\parallel. \tag{49}$$

If we used the approximate expression of the Lie generator, $g_1 \simeq g_1^{\bar{\mu}}$, as is often the case with most of the gyrokinetic analyses, the density equation is reduced to

$$n = \int (\bar{F} + g_1^{\bar{\mu}} \partial_{\bar{\mu}} \bar{F})\frac{B_\parallel^*}{m}\delta(\mathbf{X} + \rho \hat{\mathbf{a}} - \mathbf{x}) d^3 X d\Theta d\mu dV_\parallel. \tag{50}$$

Although we introduce this approximation to obtain the equilibrium density equation under the same condition as Qin's study, we can obtain more rigorous perturbation density including $g_1^{\mathbf{X}}$ such as Hahm's expression.[13] Using the partial integral for $\bar{\mu}$ and assuming that the spatial scale of the equilibrium potential is much larger than the gyroradius, we can obtain the reduced expression

$$n \simeq N + \frac{1}{\Omega^2}\nabla_\perp \cdot (U_\parallel \nabla_\parallel \mathbf{D}) + N_\mathrm{p}, \tag{51}$$

where we define the gyrocenter density N, the parallel velocity $U_\parallel$, and the polarization density N_p due to the potential perturbation as

$$N \equiv \int \bar{F}\frac{B_\parallel^*}{m}\delta(\mathbf{X}+\rho\hat{\mathbf{a}}-\mathbf{x})d^3Xd\Theta d\mu dV_\parallel, \tag{52}$$

$$U_\parallel \equiv \int V_\parallel \bar{F}\frac{B_\parallel^*}{m}\delta(\mathbf{X}+\rho\hat{\mathbf{a}}-\mathbf{x})d^3Xd\Theta d\mu dV_\parallel, \tag{53}$$

$$N_\mathrm{p} \equiv \int \frac{q^2}{m\Omega}\tilde{\phi}_1\frac{\partial \bar{F}}{\partial \bar{\mu}}\frac{B_\parallel^*}{m}\delta(\mathbf{X}+\rho\hat{\mathbf{a}}-\mathbf{x})d^3Xd\Theta d\mu dV_\parallel. \tag{54}$$

The density equation for Qin's equilibrium velocity is given by

$$n_\mathrm{Qin} = N + \frac{1}{\Omega^2}\nabla_\perp \cdot [(N\mathbf{D}_\mathrm{Qin} + U_\parallel\hat{\mathbf{b}})\cdot\nabla\mathbf{D}_\mathrm{Qin}] + N_\mathrm{p}. \tag{55}$$

The term proportional to $\mathbf{D}\cdot\nabla\mathbf{D}$ has been canceled also in the equation of the density. This relation gives the Poisson equation and thus we have obtained the whole set of the gyrokinetic equations.

V. NUMERICAL COMPARISONS

In this section, numerical verifications are given to confirm the advantages of the present equilibrium drift velocity. We solve the gyrokinetic equations of motion (45)–(48) and compare the solutions with that of the full-kinetic equations and the previous gyrokinetic equations for three kinds of potential profiles. In order to compare the solutions between those of gyrokinetics and full kinetics, the relation between the guiding-center and gyrocenter coordinate systems

$$\bar{Z}^i = Z^i + g_1^i. \tag{56}$$

The relation between the particle and guiding-center coordinate systems is given by Eqs. (1) and (2). Although approximations in the derivation of Eq. (51) are commonly used for the Lie generators in the calculations of coordinate transformations, its exact calculation yields small but not negligible improvement in solutions in the case of strong electric fields. Therefore, we use the complete expressions of the Lie generators, $g_1^{\mathbf{X}}$, g_1^{θ}, g_1^{μ}, and $g_1^{V_\parallel}$ in our numerical calculation code.

First, we examine the particle trajectories for the potential $\phi_0 = -Ey$ and $\phi_1 = 0$. The solution of the new equilibrium drift velocity, Eq. (18), for the uniform electric field is given by a simple $\mathbf{E}\times\mathbf{B}$ drift velocity $\mathbf{D} = E/B\hat{\mathbf{x}}$ for $\mathbf{B} = B\hat{\mathbf{z}}$. Therefore, the equations of motion for the previous definition of $\mathbf{D} = \mathbf{D}_\mathrm{Qin}$ and our definition coincide with each other for the uniform electric field. We solved the full-kinetic equations and the gyrokinetic equations and plotted the particle position and the gyrocenter position in Fig. 1. The initial position and velocity used in solving the gyrokinetic equations are determined from those of the full-kinetic calculation through the coordinate transformation, $\bar{\mathbf{X}} = \mathbf{x} - \rho\hat{\mathbf{a}} + g_1^{\mathbf{X}}$. The last term

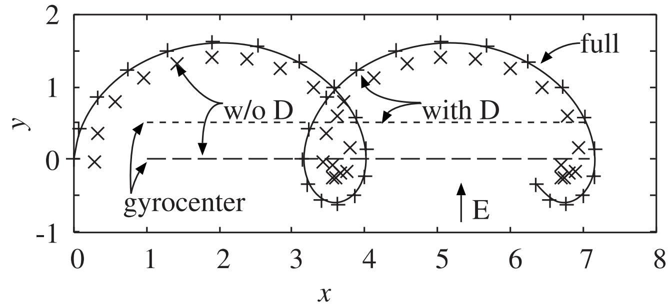

FIG. 1. Comparisons of particle trajectories calculated from the full-kinetic gyrokinetic equations with and without $\mathbf{D}$. The curve labeled "full" represents the particle orbit calculated from the full-kinetic equations of motion. The plus and cross marks correspond to the solutions of the gyrokinetic equations with and without $\mathbf{D}$, respectively. The gyrocenter orbits with and without $\mathbf{D}$ are shown as dashed and dotted lines, respectively.

$g_1^{\mathbf{X}}$ comes from the Lie transformation between the guiding-center and the gyrocenter coordinates. It represents a correction of the gyrocenter position related to the perturbation generated from the nonuniformity of the equilibrium potential and the particle gyration. The solutions of the gyrokinetic equations are transformed inversely to the particle positions, $\mathbf{x} = \bar{\mathbf{X}} + \rho\hat{\mathbf{a}} - g_1^{\bar{\mathbf{X}}}$, and plotted. We note that the Lie generator g_1 vanishes and can be ignored in the case of the uniform electric field because the reference frame moves with $\mathbf{E}\times\mathbf{B}$ drift velocity and cancels the electric field. In the case of general potential profiles, however, the Lie generator g_1 should be calculated numerically.

The gyrating curve labeled "full" in Fig. 1 represents the particle trajectory calculated from the full-kinetic equations. The plus and cross marks represent the particle positions calculated from the gyrokinetic equations with and without $\mathbf{D}$, respectively. Although the equations without $\mathbf{D}$, or $\mathbf{D}=0$, involve the equilibrium potential ϕ_0 in the calculation, they assume the ordering $|\hat{\mathbf{b}}\times\nabla\phi_0|/B_0 \ll v_\mathrm{t}$ and may yield large errors for strong electric fields. The effectiveness of the employment of the equilibrium drift velocity $\mathbf{D}$ for strong electric fields is confirmed by the fact that the particle positions with $\mathbf{D}$ are just on the curve of the full-kinetic solution, while those of $\mathbf{D}=0$ are not. The upper and lower horizontal lines in Fig. 1 represent the trajectories of the gyrocenter with and without $\mathbf{D}$, respectively. The difference in the gyrocenter positions is caused by the modification of the velocity space. Since the velocity in the gyrokinetics with $\mathbf{D}$ is defined in the frame moving with the velocity $\mathbf{D}$, the gyrocenter position shifts along the electric field. The amount of the shift is given by $\hat{\mathbf{b}}\times\mathbf{D}/\Omega = E/B\Omega\hat{\mathbf{y}}$ and corresponds to the polarization due to the equilibrium electric field.

Secondly, we use a potential profile with circular contours, $\phi_0(r) = -Er$. A particle drifts along the contour to the clockwise direction for a positive E. The solution of Eq. (18) for this potential is given by

$$\mathbf{D} = \frac{2}{1+\sqrt{1-4C}}\hat{\mathbf{b}}\times\frac{\nabla\phi_0}{B}, \tag{57}$$

$$C \equiv -\frac{|\hat{\mathbf{b}}\times\hat{\mathbf{r}}|^2}{rB\Omega}\frac{d\phi_0}{dr} = \frac{E}{rB\Omega},$$

where we denote the radius and the radial unit vector by $r=\sqrt{x^2+y^2}$ and $\hat{\mathbf{r}}=(x\hat{\mathbf{x}}+y\hat{\mathbf{y}})/r$, respectively. This velocity is larger (smaller) than that of Qin's expression $\mathbf{D}_{\mathrm{Qin}}\equiv\hat{\mathbf{b}}\times\nabla\phi_0/B$ for positive (negative) E by a factor of $2/(1+\sqrt{1-4C})$ and the polarization shift $\hat{\mathbf{b}}\times\mathbf{D}/\Omega$ is also larger (smaller). In order to confirm the accuracy of the equation in the nonuniform electric field, the conservation of the energy is examined. There are two expressions for the energy according to the coordinate systems,

$$H_{\mathrm{f}}(\mathbf{x},\mathbf{v}) = \frac{m}{2}v^2 + q\phi_0, \tag{58}$$

$$H_{\mathrm{g}}(\bar{\mathbf{X}},\bar{\mu},\bar{V}_\parallel) = \frac{m}{2}\bar{V}_\parallel^2 + B\bar{\mu} + \frac{m}{2}D^2 + q\langle\phi_0\rangle. \tag{59}$$

The former is the Hamiltonian on the particle coordinate system and represents the energy of the particle at the phase-space position $(\mathbf{x},\mathbf{v})$. The latter expression, H_{g}, is the gyrokinetic Hamiltonian on the gyrocenter coordinate system $(\bar{\mathbf{X}},\bar{\Theta},\bar{\mu},\bar{V}_\parallel)$. Since the description of dynamics consists of the definition of the coordinate system and the equations of motion, or time derivative of the coordinate variables, the Hamiltonian becomes an invariant if and only if the corresponding equations of motion are employed. Thus, the full-kinetic equation conserves the particle energy H_{f} but not the gyrokinetic energy H_{g}, and *vice versa* for the gyrokinetic equations. The reason for no conservation is that the gyrokinetic 1-form and the coordinate transformation between the particle and the gyrocenter involve truncation errors through the Taylor expansions with respect to ϵ. Therefore, the degree of energy conservation is a suitable criterion to evaluate the accuracy of the equations.

We employ two combinations of the energy expressions and equations of motion. One is the "proper" pair of the particle energy, H_{f}, and the full-kinetic equations of motion, and also H_{g} and the gyrokinetic equations. In this case, the energy is conserved rigorously. From the numerical comparison with the full-kinetic one, the consistency of the gyrokinetic equations is examined. The other combination, i.e., H_{f} and the gyrokinetic equations, is useful to examine the accuracy of the equations of motion and the coordinate transformations used in the calculation of H_{f} from the gyrokinetic coordinate variables. Although the time evolution of the energy is not stationary, it does not have a secular variation but oscillates with the gyrofrequency and its harmonics. The amplitude of the oscillation is employed as the criterion of the accuracy.

We solve the full-kinetic and the gyrokinetic equations numerically and plot three kinds of energy values in Fig. 2. First is the particle energy H_{f} calculated from the full-kinetic solution. It is shown as a solid horizontal line labeled H_{f} in the figure. Second is the gyrokinetic energy H_{g} calculated from the gyrokinetic solution. The dashed and dotted horizontal lines labeled H_{g} represent the gyrokinetic energy for our equilibrium drift velocity field $\mathbf{D}$ and that of Qin, respectively. Since the gyrokinetic energy is constant in time, we confirm that the gyrokinetic equations derived here are consistent with the 1-form. From the comparison of the deviation of the gyrokinetic energy from the full-kinetic energy, a significant improvement of the accuracy on the energy is also confirmed, i.e., $|H_{\mathrm{g}}^{\mathrm{Kawamura}}-H_{\mathrm{f}}|<|H_{\mathrm{g}}^{\mathrm{Qin}}-H_{\mathrm{f}}|$. The third energy value is the particle energy H_{f} calculated from the gyrokinetic solution. Since H_{f} is a function on the particle coordinate system, it is evaluated with the particle coordinate variables transformed from the gyrocenter coordinate variables. The dashed and dotted oscillatory curves correspond to the solutions obtained from the equations with $\mathbf{D}$ and $\mathbf{D}_{\mathrm{Qin}}$, respectively. Since the amplitude of the energy oscillation for the present $\mathbf{D}$ is reduced by a factor of 3 compared with that for the previous $\mathbf{D}_{\mathrm{Qin}}$, we confirm that the refinement of the gyrokinetic equations has been achieved by the new equilibrium velocity.

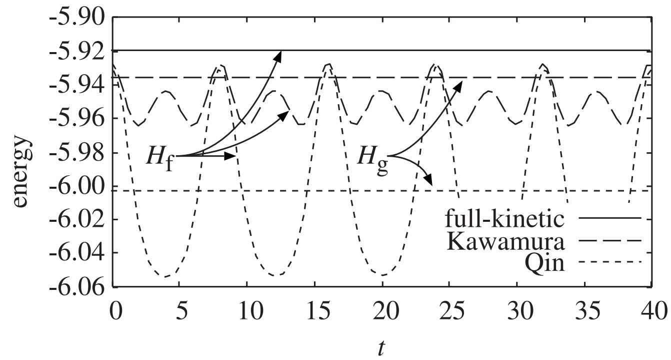

FIG. 2. Time evolution of the particle energy H_f, Eq. (58), and the gyrokinetic energy H_{g}, Eq. (59). They are calculated from the full-kinetic equations (solid) and the gyrokinetic equations with the equilibrium velocity field $\mathbf{D}$ (dashed) and $\mathbf{D}_{\mathrm{Qin}}$ (dotted).

In order to study the dependence of the error on the various plasma parameters, we plot the standard deviations of the energy oscillation for various E, B, v_{t}, and the initial velocity in Fig. 3. The standard deviation σ is normalized by the perpendicular energy $B\mu$ and can be interpreted as a relative error. The horizontal axis represents the $\mathbf{E}\times\mathbf{B}$ drift velocity normalized by the thermal velocity. Figures 3(a)–3(c) correspond to three kinds of initial positions of the particle, $r/\rho_{\mathrm{t}}=25$, 100, and 400, respectively. The cross, triangle, and circle marks correspond to the gyrokinetic equations with $\mathbf{D}=0$, the improved equations of Qin, and the equations derived here, respectively. The broad distributions in $\sigma/B\mu$, especially for the equations with $\mathbf{D}=0$, are caused by the thermal spread in the perpendicular velocity space, which has a Maxwellian distribution. The relative error of the equations with $\mathbf{D}=0$ does not depend on the initial position, which corresponds to the curvature of the potential contour in this case, while the error of the improved equations with $\mathbf{D}$ decreases for smaller curvature. This fact indicates that the error in the equations with $\mathbf{D}=0$ depends on the electric field strength, while that of the equations with $\mathbf{D}$ depends on the second derivative of ϕ_0 rather than the first derivative or the strength of the electric field. This tendency agrees with the observation on the conservation of the magnetic moment in Sec. III. The relative error

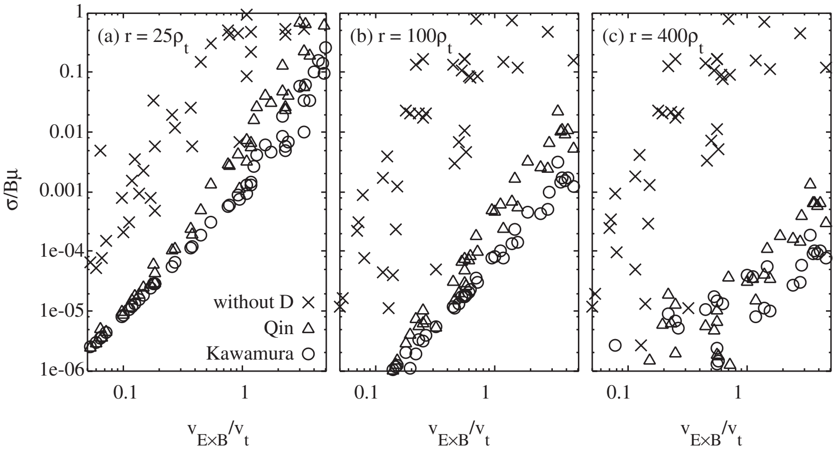

FIG. 3. Standard deviations of energy oscillations for the $\mathbf{E}\times\mathbf{B}$ drift velocity. (a), (b), and (c) correspond to three kinds of the initial positions of the particles, r/ρ_t=25, 100, and 400, respectively. The cross, triangle, and circle marks correspond to the gyrokinetic equations with $\mathbf{D}=0$, the improved equations of Qin, and the equations derived here, respectively.

with the present $\mathbf{D}$ is roughly estimated from Fig. 3 as $\Delta \simeq (v_{\mathbf{E}\times\mathbf{B}}/v_t)^2/2(r/\rho_t)^2 = (E/rB\Omega)^2/2 \simeq (\nabla^2\phi/B\Omega)^2/2$. From the comparisons between Qin's previous solutions (triangle marks in Fig. 3) and ours (circle marks), the reduction of the error is achieved when the $\mathbf{E}\times\mathbf{B}$ drift velocity exceeds approximately 1/10 of the thermal velocity. The maximum reduction around 1/10 is achieved when $v_{\mathbf{E}\times\mathbf{B}} > v_t$.

Finally, we examine the energy oscillation for a more general potential profile. Since there is no analytic expression for the solution of Eq. (18), numerical calculations are required for general profiles. We used a simple recurrence equation to solve the equilibrium drift velocity,

$$\mathbf{D}_0 = \hat{\mathbf{b}} \times \frac{\nabla \phi_0}{B}, \tag{60}$$

$$\mathbf{D}_{n+1} = \hat{\mathbf{b}} \times \left(\frac{\nabla \phi_0}{B} + \mathbf{D}_{n+1} \cdot \frac{\nabla \mathbf{D}_n}{\Omega} \right), \tag{61}$$

where the gradient of the vector $\mathbf{D}_n(\mathbf{X})$ is calculated as

$$\nabla \mathbf{D}_n(\mathbf{X}) = \sum_i \hat{\mathbf{z}}^i \frac{\mathbf{D}_n(\mathbf{X}+\delta\hat{\mathbf{z}}^i) - \mathbf{D}_n(\mathbf{X}-\delta\hat{\mathbf{z}}^i)}{2\delta}. \tag{62}$$

The small quantity δ is chosen to be much smaller than the scale length of the potential ϕ_0. Sufficient accuracy for general purposes can be obtained by two or three iterations. We used an equilibrium potential with elliptic contours as an example, $\phi_0 = -E\sqrt{4x^2+y^2}$. The time evolution of the particle energy calculated from the gyrokinetic equations with $\mathbf{D}$ and $\mathbf{D}_{\mathrm{Qin}}$ is shown in Fig. 4. The period, $T=92$, equals a 1-cycle of the rotation along the contour. At the beginning of the calculation, the particle is located on the x axis, $(x,y) = (l,0)$, and drifts clockwise to $(0,-2l)$ at $t=T/4=23$. The slow variation of the envelope is caused by the spatial difference of the curvature of the potential contour. The amplitude of the energy oscillation is reduced by a factor of 3 also in the potential with the elliptic contour.

VI. CONCLUSIONS

Refinement of the equilibrium drift velocity in the gyrokinetic theory has been proposed for edge plasmas with large $\mathbf{E}\times\mathbf{B}$ flow shears. An equilibrium velocity field $\mathbf{D}$ is introduced in the coordinate transformations, Eqs. (1) and (2), to decouple the drift motion and gyration of a charged particle in the zeroth-order dynamics in Eq. (7). We investigated the effects of the velocity $\mathbf{D}$ on the zeroth-order equation of motion, especially on the magnetic moment μ, and obtained the practically most accurate expression of $\mathbf{D}$, Eq. (18).

Using the standard procedures of Lie perturbation analysis, we obtained the general expressions of the Lie generator, Eq. (29), and the gyrokinetic 1-form, Eq. (32), up to the first order. As a limiting case, the electrostatic gyrokinetic equations of motion, Eqs. (45)–(48), and the particle density, Eq. (51), were derived. It was confirmed that a term proportional to $\mathbf{D}\cdot\nabla\mathbf{D}$ in the gauge function, Eq. (38), used by Qin was canceled through the refinement of $\mathbf{D}$ in our gauge function, Eq. (36). This fact indicates that our modification in $\mathbf{D}$ reduces the error involved in the zeroth-order dynamics.

The advantages of our formulation were also confirmed in the numerical verifications in Sec. V. The accuracy of the equations of motion was estimated through the conservation of the particle energy calculated from the gyrocenter coordinate variables $\bar{Z} = (\bar{\mathbf{X}}, \bar{\Theta}, \bar{\mu}, \bar{V}_\parallel)$. When the $\mathbf{E}\times\mathbf{B}$ drift velocity is comparable to the thermal velocity, the oscillatory behavior of the energy due to the truncations at second order was reduced up to 1/10 in its standard deviation compared with the previous formulation by Qin.

From the analytic investigation and the numerical verifications, it has been confirmed that the refinement of the

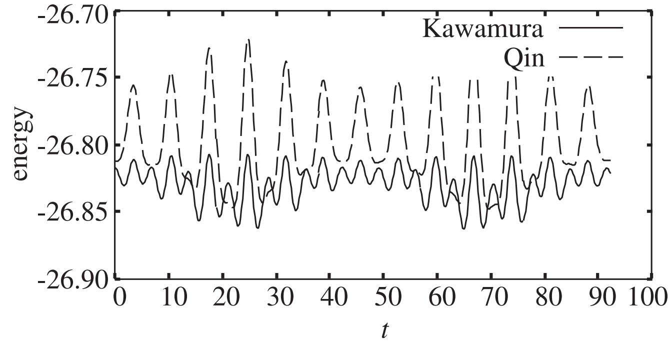

FIG. 4. Time evolution of the particle energy. The solid and dashed lines correspond to the solution of our gyrokinetic equations and those of Qin, respectively.

equilibrium velocity $\mathbf{D}$ succeeds in obtaining more accurate equations of motion and gyrocenter coordinate. The general expressions of the charge and current densities were formulated and the approximate density equation (51) for the electrostatic potential was also obtained. Our formulation is, however, based on the single-particle 1-form, and thus the self-consistency for collective dynamics, or a plasma, is not fully ensured by itself. The self-consistency, e.g., conservation of the plasma energy, for the gyrokinetics without large equilibrium $\mathbf{E}\times\mathbf{B}$ flow has been confirmed by the field theory.[26] The self-consistency is essential not only for the theoretical completeness but for the numerical simulation as a guarantee of the conservation of energy and momentum. The application of the field theory to the gyrokinetic theory with the strong $\mathbf{E}\times\mathbf{B}$ flow will be an important topic of further studies.

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[1]B. I. Cohen, T. J. Williams, A. M. Dimits, and J. A. Byers, Phys. Fluids B **5**, 2967 (1993).
[2]G. Rewoldt, M. A. Beer, M. S. Chance, T. S. Hahm, Z. Lin, and W. M. Tang, Phys. Plasmas **5**, 1815 (1998).
[3]M. Maccio, J. Vaclavik, and L. Villard, Phys. Plasmas **8**, 895 (2001).
[4]J. E. Kinsey, R. E. Waltz, and J. Candy, Phys. Plasmas **12**, 062302 (2005).
[5]B. Scott, Plasma Phys. Controlled Fusion **48**, A387 (2006).
[6]J. A. Heikkinen, S. Henriksson, S. Janhunen, T. P. Kiviniemi, and F. Ogando, Contrib. Plasma Phys. **46**, 490 (2006).
[7]X. Q. Xu, Z. Xiong, M. R. Dorr, J. A. Hittinger, K. Bodi, J. Candy, B. I. Cohen, R. H. Cohen, P. Colella, G. D. Kerbel, S. Krasheninnikov, W. M. Nevins, H. Qin, T. D. Rognlien, P. B. Snyder, and M. V. Umansky, Nucl. Fusion **47**, 809 (2007).
[8]A. J. Brizard, "Nonlinear gyrokinetic tokamak physics," Ph.D. thesis, Princeton University (1990).
[9]H. Qin, "Gyrokinetic theory and computational methods for electromagnetic perturbations in tokamaks," Ph.D. thesis, Princeton University (1998).
[10]A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
[11]R. G. Littlejohn, Phys. Fluids **24**, 1730 (1981).
[12]A. J. Brizard, Phys. Plasmas **2**, 459 (1995).
[13]T. S. Hahm, Phys. Plasmas **3**, 4658 (1996).
[14]H. Qin, Contrib. Plasma Phys. **46**, 477 (2006).
[15]P. H. Rutherford and E. A. Frieman, Phys. Fluids **11**, 569 (1968).
[16]J. B. Taylor and R. J. Hastie, Plasma Phys. **10**, 479 (1968).
[17]T. M. Antonsen, Jr., and B. Lane, Phys. Fluids **23**, 1205 (1980).
[18]P. J. Catto, W. M. Tang, and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).
[19]E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).
[20]H. Sugama and W. Horton, Phys. Plasmas **5**, 2560 (1998).
[21]R. G. Litlejohn, Phys. Rev. A **38**, 6034 (1988).
[22]A. Baños, Jr., J. Plasma Phys. **1**, 305 (1965).
[23]H. Qin and W. M. Tang, Phys. Plasmas **11**, 1052 (2004).
[24]T. S. Hahm, Phys. Fluids **31**, 2670 (1988).
[25]Y. Idomura, T.-H. Wakanabe, and H. Sugama, C. R. Phys. **7**, 650 (2006).
[26]H. Sugama, Phys. Plasmas **7**, 466 (2000).